Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Mathematics and Mechanics 71 (2007) 247–257
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

The effect of vertical vibration on the onset of thermocapillary convection in a horizontal liquid layer ☆

S.M. Zen'kovskaya, V.A. Novosyadlyi, A.L. Shleikel'

Rostov-on-Don, Russia

Received 12 January 2006

Abstract

The effect of vertical vibration on the onset of Marangoni convection in a horizontal layer of a viscous incompressible uniform liquid with a free surface and a hard (solid) or soft (impermeable and stress-free) wall is investigated. In the case of harmonic vibration, a dispersion relation is constructed in explicit form using continued fractions. From this, equations are obtained for determining the critical values of the parameters for all three main types of loss of stability. Neutral curves of the monotonic and oscillatory instability are constructed, for fixed frequency and amplitude of the vibration, in the form of a graph of the Marangoni number against the wave number. The regions of parametric resonances, corresponding to synchronous and subharmonic modes are determined. The frequency values for which a high-frequency asymptotic form is reached are obtained. The long-wave Marangoni oscillatory instability is investigated, and it is shown that in this case the Marangoni numbers are negative and depend only on the Prandtl and Biot numbers.

The problem of the effect of vertical vibration on the onset of convection in a region with a solid boundary was investigated in Ref. 1 in the case of high-frequency vibration and finite amplitude of the velocity. The averaging method was employed, which enables the unknowns to be separated into slow and fast components, enables the fast component to be expressed in terms of the slow component, and enables average equations to be obtained for the latter. Later, a number of papers were published which analysed the average equations of convection, and the investigation of vibration convection became a separate area of the theory of convective instability. Here both gravitational convection and convection in zero gravity were considered. A review of publications on vibration convection can be found in Ref. 2. The first experiments to investigate the phenomena of parametric resonance and dynamic instability in gravitational thermal convection were described in Ref. 3, and a review of the theoretical and experimental results on convection in modulated fields is also given there. The effect of high-frequency vibration on the onset of thermocapillary convection is considered in Refs. 4–8. There are not many theoretical papers on convection in which parametric actions of finite frequency are considered, due to mathematical difficulties. In the case of a small modulation in systems close to parametric resonance, perturbation methods are employed.[9–11] For finite amplitudes of the modulation, either grid methods or the Galerkin–Kantorovich methods are used.[12–15] The effect of thermal modulation on the onset of Marangoni–Bénard convection in a horizontal layer was also investigated in Ref. 16, where a considerable list of papers on parametric actions on thermocapillary convection are listed.

☆ *Prikl. Mat. Mekh.* Vol. 71, No. 2, pp. 277–288, 2007.
E-mail address: zenkov@math.rsu.ru (S.M. Zen'kovskaya).

doi:10.1016/j.jappmathmech.2007.06.008

In the case of harmonic vibration, when the equilibrium solution allows of a constant vertical temperature gradient, in research on gravitational convection by Yudovich and his students (Refs. 17–19),[a] the method of continued fractions was used to calculate the critical values of the parameters.[20] This method was also employed when considering the stability of steady flows under conditions of spatial periodicity.[21,22]advantage of the method is that it does not require constraints on the parameters and can easily be realized. In problems with free boundaries or interfaces, an approach was developed in Ref. 23, which also enabled a dispersion equation to be constructed in the form of continued fractions. This method is also employed in the present paper, the purpose of which is not only to obtain qualitative results, but also to demonstrate effective methods of solving similar problems.[b]

Particular attention is devoted here to cases of high and finite frequencies. It is well known that, at low modulation frequencies, no numerical methods, as a rule, work well; in such situations, asymptotic methods (most often of all the WKB method[24]) are used, which were employed for the first time in convection problems by Markman and Yudovich in Ref. 17.

1. Formulation of the problem

Consider the motion in a layer of a viscous incompressible uniform heat-conducting liquid, bounded above by a free surface $x_3 = \xi(x_1, x_2, t)$, and below by a hard (solid) or "soft" wall $x_3 = H$. We mean by a "soft" wall an inpermeable stress–free surface. We will take into account the deformability of the free boundary and the presence of surface-tension forces with a coefficient $\tilde{\alpha} = \alpha_0 - \alpha_T(T - T_0)$, which depend linearly on the temperature. We will assume that the wall executes vertical vibration, given by the relation $x_3 = \tilde{a} f(\tilde{\omega} t)$, where f is a 2π-periodic function with zero mean, $\tilde{\omega}$ is the frequency and $\tilde{a} = \tilde{a}(\tilde{\omega})$ is the vibration amplitude.

In a Cartesian system of coordinates, rigidly connected with the vibrating wall, the convection equations have the form

$$\begin{aligned} &\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p + \nu\Delta\mathbf{v} + g(t)\boldsymbol{\gamma}, \quad \operatorname{div}\mathbf{v} = 0 \\ &\frac{\partial T}{\partial t} + (\mathbf{v}, \nabla T) = \chi\Delta T \end{aligned} \tag{1.1}$$

Here $\boldsymbol{v} = (v_1, v_2, v_3)$ is the relative velocity of the liquid, p is the pressure, T is the temperature, measured from a certain value T_0, ρ_0 is the liquid density, ν and χ are the kinematic viscosity coefficient and thermal conductivity, and $\boldsymbol{\gamma}$ is the unit vector of the downward directed x_3 axis (the quantities x_1 and x_3 will sometimes be denoted by x and z). The origin of coordinates is chosen on the unperturbed free surface. The function $g(t) = g_0 - \tilde{a}\tilde{\omega}^2 f''(\tilde{\omega} t)$ is the variable acceleration due to gravity. If the free boundary is situated below (an inverted layer), we must replace $\boldsymbol{\gamma}$ by $-\boldsymbol{\gamma}$. We will assume that above the liquid there is a gas whose density is negligibly small, while the temperature and pressure are constant. The following boundary conditions must be satisfied on the free boundary $x_3 = \xi(x_1, x_2, t)$

$$\begin{aligned} &(\mathbf{v}, l) = \frac{\partial \xi}{\partial t}; \quad l = \left(-\frac{\partial \xi}{\partial x_1}, -\frac{\partial \xi}{\partial x_2}, 1\right) \\ &\tau_{ik} n_k - p n_i = -\tilde{\alpha}\kappa n_i - \frac{\partial \tilde{\alpha}}{\partial x_i} + \frac{\partial \tilde{\alpha}}{\partial x_k} n_k n_i \\ &\tau_{ik} = \nu\rho_0\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), \quad i, k = 1, 2, 3 \\ &k_1 \frac{\partial T}{\partial n} - b_1 T = \delta_1 \end{aligned} \tag{1.2}$$

[a] See also: Yudovich V I, Belen'kaya L Kh. Numerical investigation of the onset of convection in a binary mixture under the action of time-periodic external forces. Rostov-on-Don, 1981. Deposited at VINITI 4 January 1981, No.1570-81.

[b] For a preliminary account see Zen'kovskaya S M, Shleikel' A L, Novosyadlyi V A. The effect of vertical vibration of finite frequency on the onset of thermocapillary convection in a horizontal layer. Rostov-on-Don, 2003. Deposited at VINITI 27 July 2003. No. 1440-2003.

Here l is the normal to the free boundary, $\mathbf{n} = l/|l|$ is its unit vector, τ_{ik} are the components of the viscous stress tensor, and κ is the mean curvature, which is calculated from the formula

$$\kappa = \frac{(1+\xi_{x_1}^2)\xi_{x_2x_2} + (1+\xi_{x_2}^2)\xi_{x_1x_1} - 2\xi_{x_1}\xi_{x_2}\xi_{x_1x_2}}{(1+\xi_{x_1}^2+\xi_{x_2}^2)^{3/2}}$$

We will henceforth consider liquid flows that are periodic in x_1 and x_2 with periods L_1 and L_2. We will correspondingly assume that the velocity $\boldsymbol{v}$, the pressure p, the temperature T and the function ξ in the equation of the free boundary are periodic in x_1 and x_2. Moreover, we will assume that the mean thickness of the layer is specified and equal to H, so that

$$\frac{1}{L_1L_2}\int_0^{L_1}\int_0^{L_2}\xi(x_1, x_2, t)dx_1dx_2 = 0$$

This condition, in particular, means that, when investigating the stability, perturbations due to a change in the average amount of liquid in the layer, are eliminated.

We will specify the following boundary conditions on the hard wall

$$x_3 = H\colon \boldsymbol{v} = 0, \quad k_2\frac{\partial T}{\partial x_3} + b_2T = \delta_2 \tag{1.3}$$

If the wall is soft, the boundary conditions on it have the form

$$x_3 = H\colon v_3 = 0, \quad \frac{\partial v_j}{\partial x_3} + \frac{\partial v_3}{\partial x_j} = 0, \quad j = 1, 2; \quad k_2\frac{\partial T}{\partial x_3} + b_2T = \delta_2 \tag{1.4}$$

2. The quasi-equilibrium solution

2.1. The stability problem

We will assume that the heat-transfer conditions are chosen in such a way that problems (1.1)–(1.3) and (1.1), (1.2), (1.4) have a solution corresponding to relative equilibrium with a linear temperature profile

$$\boldsymbol{v}^0 = 0, \quad p^0 = \rho_0 g(t)z, \quad \xi^0 = 0, \quad T^0 = Az$$

Here A is the equilibrium temperature gradient.

We will investigate the stability of this solution, assuming that

$$\boldsymbol{v} = \boldsymbol{v}^0 + \mathbf{u}, \quad p = p^0 + P, \quad \xi = \xi^0 + \eta, \quad T = T^0 + \theta$$

We will change to dimensionless variables, choosing the following scales: length $\mathcal{L}$, time $\mathcal{T}$, velocity $\mathcal{L}/\mathcal{T}$, pressure $\rho_0\mathcal{L}^2/\mathcal{T}^2$ and temperature $A\mathcal{L}$. The scales $\mathcal{L}$ and $\mathcal{T}$ remain arbitrary for the present. Retaining as dimensionless variables the same notation as for the dimensional variables, we will write the system for infinitesimal perturbations in the form

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla P + \delta\Delta\mathbf{u}, \quad \operatorname{div}\mathbf{u} = 0, \quad \frac{\partial \theta}{\partial t} = \mathrm{Pr}^{-1}\Delta\theta - u_3 \tag{2.1}$$

Linearisation of boundary conditions (1.2) gives the conditions on the unperturbed free boundary

$$\begin{gathered} x_3 = 0\colon u_3 = \frac{\partial \eta}{\partial t}, \quad \frac{\partial u_j}{\partial x_3} + \frac{\partial u_3}{\partial x_j} = \frac{\mathrm{Ma}}{\mathrm{Pr}}\left(\frac{\partial \theta}{\partial x_j} + \frac{\partial \eta}{\partial x_j}\right), \quad j = 1, 2 \\ 2\delta\frac{\partial u_3}{\partial x_3} - P + C\Delta_1\eta - Q(t)\eta = 0, \quad \frac{\partial \theta}{\partial x_3} - \mathrm{Bi}(\theta + \eta) = 0 \end{gathered} \tag{2.2}$$

We will write the boundary conditions on the soft (hard) wall in the form

$$x_3 = h\colon u_3 = 0, \quad \frac{\partial u_j}{\partial x_3} + \frac{\partial u_3}{\partial x_j} = 0, \quad j = 1, 2 \quad (\mathbf{u} = 0)$$
$$\frac{\partial \theta}{\partial x_3} + B\theta = 0 \tag{2.3}$$

Problem (2.1)–(2.3) contains the following dimensionless parameters

$$\delta = \frac{\nu \mathcal{T}}{\mathcal{L}^2}, \quad \mathrm{Pr} = \frac{\mathcal{L}^2}{\chi \mathcal{T}}, \quad \mathrm{Ma} = \frac{\alpha_T A \mathcal{L}^2}{\rho_0 \nu \chi}, \quad C = \frac{\alpha_0 \mathcal{T}^2}{\rho_0 \mathcal{L}^3}, \quad \mathrm{Bi} = \frac{b_1 \mathcal{L}}{k_1}$$

$$B = \frac{b_2 \mathcal{L}}{k_2}, \quad \omega = \tilde{\omega}\mathcal{T}, \quad k = \frac{H}{\mathcal{L}}, \quad Q(t) = Q_0 - a\omega^2 f''(\omega t), \quad Q_0 = \frac{g_0 \mathcal{T}^2}{\mathcal{L}}, \quad a = \frac{\tilde{a}}{\mathcal{L}}$$

Here δ is the dimensionless viscosity, Pr is the Prandtl number, Ma is the Marangoni number, C is the dimensionless surface tension coefficient, Bi is the Biot number, B is the heat-transfer parameter (this will sometimes be written in the form $B = B_0/B_1$), ω is the dimensionless modulation frequency, h is the mean thickness of the layer, $Q(t)$ is the variable dimensionless acceleration, where Q_0 is its average part (g_0 is the acceleration due to gravity), and $-a\omega^2 f''(\omega t)$ is its modulation with amplitude $a\omega^2$ (a is the dimensionless modulation amplitude).

Further, for brevity, we will assume that the perturbations are plane: $u_2 = 0$, and the unknowns u_1, u_3, P, θ, η are independent of the variable x_2. If we consider three-dimensional modes, we obtain the same equations, except that instead of the wave number we will have the modulus of the wave vector. We will introduce the stream function $\psi(x, z, t)$, assuming $u_1 = \partial\psi/\partial z$, $u_3 = -\partial\psi/\partial x$, and we will eliminate the pressure. Separating the variable x by making the replacement

$$(\psi(x, z, t), \eta(x, t), \theta(x, z, t)) = (i\alpha\hat{\psi}(z, t), \hat{\eta}(t), \hat{\theta}(z, t))e^{i\alpha x}$$

for the amplitudes $\hat{\psi}$, $\hat{\eta}$, $\hat{\theta}$ we obtain the problem ($D = \partial/\partial z$, $F_t = \partial F/\partial t$)

$$(D^2 - \alpha^2)\hat{\psi}_t = \delta(D^2 - \alpha^2)^2\hat{\psi}, \quad \hat{\theta}_t = \mathrm{Pr}^{-1}(D^2 - \alpha^2)\hat{\theta} - \alpha^2\hat{\psi} \tag{2.4}$$

$$z = 0\colon \hat{\eta}_t = \alpha^2\hat{\psi}, \quad D^2\hat{\psi} + \alpha^2\hat{\psi} = \mathrm{MaPr}^{-1}(\hat{\theta} + \hat{\eta})$$
$$D\hat{\psi}_t - \delta(D^2 - \alpha^2)D\hat{\psi} + 2\delta\alpha^2 D\hat{\psi} - (Q(\omega t) + C\alpha^2)\hat{\eta} = 0 \tag{2.5}$$
$$D\hat{\theta} - \mathrm{Bi}(\hat{\theta} + \hat{\eta}) = 0$$

$$z = h\colon \hat{\psi} = 0, \quad D^2\hat{\psi} = 0 \quad (D\hat{\psi} = 0), \quad D\hat{\theta} + B\hat{\theta} = 0 \tag{2.6}$$

The cases of soft and hard walls differ by the second condition (2.6): on the soft wall $D^2\hat{\psi} = 0$ and on the hard wall $D\hat{\psi} = 0$.

3. Floquet solutions

3.1. The dispersion equation

We will assume that $f(\omega t)$ has a Fourier expansion: $f(\omega t) = \sum f_j e^{ij\omega t}$ (summation over j here and henceforth is from $-\infty$ to $+\infty$). We will seek a solution of (2.4)–(2.6) in the form

$$(\hat{\psi}(z, t), \hat{\eta}(t), \hat{\theta}(z, t)) = e^{\sigma t}\sum(\psi_n(z), c_n, \theta_n(z))e^{in\omega t} \tag{3.1}$$

where σ is the Floquet exponent – an unknown complex number, which defines the behaviour of the perturbations with time. For the unknowns $\psi_n(z)$, c_n, $\theta_n(z)$ we obtain an infinite system of ordinary differential equations

$$\sigma_n L\psi_n = \delta L^2\psi_n, \quad \sigma_n\theta_n = \mathrm{Pr}^{-1}L\theta_n - \alpha^2\psi_n; \quad n = 0, \pm 1, \ldots \tag{3.2}$$

$$z = 0{:}\ \alpha^2\psi_n = \sigma_n c_n, \quad D^2\psi_n + \alpha^2\psi_n = \mathrm{MaPr}^{-1}(\theta_n + c_n) \tag{3.3}$$

$$z = 0{:}\ \sigma_n D\psi_n + 3\mu D\psi_n - \delta D^3\psi_n - \Omega^2\alpha^{-1}c_n - 2q\alpha^{-1}\sum f_j c_k = 0, \quad j+k = n \tag{3.4}$$

$$z = 0{:}\ D\theta_n - \mathrm{Bi}(\theta_n + c_n) = 0 \tag{3.5}$$

$$z = h{:}\ \psi_n = 0, \quad D^2\psi_n = 0\ (D\psi_n = 0), \quad D\theta_n + B\theta_n = 0 \tag{3.6}$$

Here $\sigma_n = \sigma + in\omega$, $\mu = \delta\alpha^2$, $\Omega^2 = C\alpha^3 + Q_0\alpha$, $q = a\omega^2\alpha/2$, $L = D^2 - \alpha^2$.

Now, for each value of n we express the unknown functions $\psi_n(z)$ and $\theta_n(z)$ in terms of c_n, by solving problem (3.2), (3.3), (3.5), (3.6), and then substitute them into boundary condition (3.4). As a result, we obtain an infinite system of linear algebraic equations for the coefficient c_n, which we will write in the form

$$M_n c_n = -2q\sum f_j c_k, \quad n = 0, \pm 1, \ldots, \quad j+k = n \tag{3.7}$$

Equating the determinant of this system to zero, we obtain a transcendental equation, which can be used both to find the Floquet spectrum of σ for fixed values of the remaining parameters and to determine the critical values of the parameters and to construct neutral curves.

Further, we will consider the case when $f(\omega t) = \cos\omega t$, so that in the Fourier expansion $f_{-1} = f_1 = 1/2$, while the remaining coefficients are equal to zero. Then, system (3.7) becomes a three-diagonal system

$$M_n c_n = -q(c_{n-1} + c_{n+1}), \quad n = 0, \pm 1, \ldots \tag{3.8}$$

The expressions for M_n will be different for soft and hard walls.

3.2. A soft wall

When $\sigma_n \neq 0$ and $\mathrm{Pr}\delta \neq 1$, we obtain

$$M_n = \Omega^2 + \sigma_n[-(\sigma_n + 2\mu)\kappa_n \operatorname{cth}\alpha h + 2\mu\beta_n\alpha^{-1}(\kappa_n + 1)\operatorname{cth}\beta_n h] \tag{3.9}$$

Here we have used the following notation

$$\begin{gathered}
\kappa_n = -\frac{\sigma_n(\sigma_n + 2\mu) + \mathrm{MaPr}^{-1}\mu(m_n - r_n)}{\sigma_n^2 + \mathrm{MaPr}^{-1}\mu(m_n + l_n)} \\
l_n = \frac{\operatorname{cth}\alpha h d_1(\alpha) + d_2(\alpha)}{d_0}, \quad m_n = \frac{\operatorname{cth}\beta_n h d_1(\beta_n) + d_2(\beta_n)}{d_0(1 - \mathrm{Pr}\delta)} \\
r_n = \frac{\gamma_n k_2(\gamma_n)}{d_0}, \quad \beta_n = \sqrt{\alpha^2 + \delta^{-1}\sigma_n} \quad \gamma_n = \sqrt{\alpha^2 + \mathrm{Pr}\sigma_n} \\
d_0 = \mathrm{Bi}k_1(\gamma_n) + k_2(\gamma_n)\gamma_n, \quad d_1(\varepsilon) = k_1(\gamma_n)\varepsilon - k_1(\varepsilon)\gamma_n, \quad d_2(\varepsilon) = \gamma_n(k_2(\varepsilon) - k_2(\gamma_n)) \\
k_1(\varepsilon) = \varepsilon\operatorname{ch}\varepsilon h + B\operatorname{sh}\varepsilon h, \quad k_2(\varepsilon) = \varepsilon\operatorname{sh}\varepsilon h + B\operatorname{ch}\varepsilon h
\end{gathered} \tag{3.10}$$

When $\sigma_n = 0$, the formula for M_n takes the form

$$
\begin{aligned}
M_n &= \Omega^2 - \frac{8\mu\alpha^3 h \mathrm{MaPr}^{-1}(\alpha \mathrm{sh}\alpha h + B\mathrm{ch}\alpha h)}{\alpha a_1 + Bb_1} \\
a_1 &= \mathrm{Ma}(\alpha^2 h^2 - \mathrm{sh}^2\alpha h)\mathrm{sh}\alpha h + 8\alpha \mathrm{sh}^2\alpha h(\mathrm{Bi}\,\mathrm{ch}\alpha h + \alpha \mathrm{sh}\alpha h) \\
b_1 &= \mathrm{Ma}(2\alpha^2 h^2 \mathrm{ch}\alpha h - \mathrm{ch}\alpha h \mathrm{sh}^2\alpha h - \alpha h \mathrm{sh}\alpha h) + 8\alpha \mathrm{sh}^2\alpha h(\mathrm{Bi}\,\mathrm{sh}\alpha h + \alpha \mathrm{ch}\alpha h)
\end{aligned}
\tag{3.11}
$$

In particular, when $\sigma = 0$ we can use formula (3.11) to calculate M_0.

3.3. A hard wall

If $\sigma + in\omega \neq 0$, formulae (3.9) and (3.10) remain the same, except that l_n must be replaced by $\bar{l}_n$, and m_n must be replaced by $\bar{m}_n$, where

$$
\begin{aligned}
\bar{l}_n &= \frac{-s_1 d_1(\alpha) + s_0 d_2(\alpha) - (\mathrm{Pr}\delta - 1)^{-1}\alpha d_1(\beta_n)}{s_0 d_0} \\
\bar{m}_n &= -\frac{s_0 d_2(\beta_n) + s_2 d_1(\beta_n) + (\mathrm{Pr}\delta - 1)\beta_n d_1(\alpha)}{s_0 d_0} \\
s_0 &= \beta_n \mathrm{sh}\alpha h \cos\beta_n h - \alpha \mathrm{ch}\alpha h \mathrm{sh}\beta_n h \\
s_1 &= \beta_n \mathrm{ch}\alpha h \mathrm{ch}\beta_n h - \alpha \mathrm{sh}\alpha h \mathrm{sh}\beta_n h \\
s_2 &= \beta_n \mathrm{sh}\alpha h \mathrm{sh}\beta_n h - \alpha \mathrm{ch}\alpha h \mathrm{ch}\beta_n h
\end{aligned}
\tag{3.12}
$$

When $\sigma_n = 0$ we have

$$
\begin{aligned}
M_n &= \Omega^2 - \frac{8\mu\alpha^3 h^2 \mathrm{MaPr}^{-1}(\alpha \mathrm{sh}\alpha h + B\mathrm{ch}\alpha h)}{\alpha \bar{a}_1 + B\bar{b}_1} \\
\bar{a}_1 &= \mathrm{Ma}[(\alpha^3 h^3 + 2\alpha h)\mathrm{sh}\alpha h - (\alpha^2 h^2 + \mathrm{sh}^2\alpha h)\mathrm{ch}\alpha h] + \\
&+ 8\alpha(\mathrm{ch}\alpha h \mathrm{sh}\alpha h - \alpha h)(\alpha h \mathrm{sh}\alpha h + \mathrm{Bi}\,\mathrm{ch}\alpha h) \\
\bar{b}_1 &= \mathrm{Ma}(\alpha^3 h^3 \mathrm{ch}\alpha h - \mathrm{sh}^3\alpha h) + 8\alpha(\mathrm{ch}\alpha h \mathrm{sh}\alpha h - \alpha h)(\mathrm{Bi}\,\mathrm{sh}\alpha h + \alpha \mathrm{ch}\alpha h)
\end{aligned}
\tag{3.13}
$$

Note that, in passing, we have obtained expressions for the critical values of the Marangoni number and a transcendental equation for the frequency of neutral oscillations when there are no vibrations ($a = 0$ and, as a consequence, $q = 0$). System (3.8) then consists of a single equation $M_0(\sigma)c_0 = 0$. In the case of monotonic instability ($\sigma = 0$), the expression for M_0 is given by formulae (3.11) or (3.13), and from the equation $M_0 = 0$ we obtain a formula for the Marangoni number Ma, which is identical with the well-known formulae in Ref. 25. In the case of oscillatory instability $\sigma_0 = ic$, from the equation $M_0 = 0$, where M_0 has already been found from formula (3.9), we can write the Marangoni number in the form $\mathrm{Ma} = \mathrm{Ma}(c)$, and from the condition for this parameter to be real we obtain a transcendental equation for the neutral oscillation frequency c (see the preprint indicated in the second footnote).

3.4. The method of continued fractions

For tridiagonal systems we can write the dispersion relation in explicit form using continued fractions. This is described in detail in a number of papers (see, for example, Refs. 20–23). We will briefly present it here.

We rewrite system (3.8) in the following form

$$a_n c_n + c_{n-1} + c_{n+1} = 0, \quad a_n = M_n/q, \quad n = 0, \pm 1, \ldots \tag{3.14}$$

and assuming $\rho_n = c_n/c_{n-1}$, we reduce it to the form (the validity of this change has been substantiated in Ref. 17)

$$a_n + \rho_n^{-1} + \rho_{n+1} = 0 \tag{3.15}$$

Now, from Eq. (3.15), using continued fractions, we obtain two representations for ρ_n

$$\rho_n = -a_{n-1} + \cfrac{-1}{-a_{n-2} + \cfrac{-1}{-a_{n-3} + \ldots}}, \quad \rho_n = \cfrac{-1}{a_n + \cfrac{-1}{a_{n+1} + \ldots}} \tag{3.16}$$

equating which for $n = 1$ and substituting the expressions for a_n, we obtain a dispersion equation for the Floquet exponent σ

$$-M_0 - \cfrac{-q^2}{M_1 + \cfrac{-q^2}{M_2 + \ldots}} = \cfrac{-q^2}{M_{-1} + \cfrac{-q^2}{M_{-2} + \ldots}} \tag{3.17}$$

Later we will be concerned with neutral curves, for which $\mathrm{Re}\sigma = 0$ and $\mathrm{Im}\sigma = ic$. In this case, the unknowns are the critical Marangoni number Ma and the neutral oscillation frequency c.

When $\sigma = 0$ we have the symmetry relation $\overline{M_n(0)} = M_{-n}(0)$, as a result of which Eq. (3.17) is simplified and can be reduced to the real form

$$\mathrm{Re}\cfrac{-q^2}{M_1(0) + \cfrac{-q^2}{M_2(0) + \ldots}} = -\frac{M_0}{2} \tag{3.18}$$

If $\sigma = i\omega/2$, we have the equality $M_{-n}(i\omega/2) = \overline{M_{n-1}(i\omega/2)}$, and the characteristic equation, corresponding to the onset of a neutral oscillation of double period, takes the form

$$\left| M_0 + \cfrac{-q^2}{M_1 + \cfrac{-q^2}{M_2 + \ldots}} \right|^2 = q^2 \tag{3.19}$$

Hence, we have, in explicit form, characteristic equations for determining the critical values of the parameters for three fundamental transitions to secondary modes: of the same period as the fundamental mode, of double period, and also to two-frequency quasi-periodic modes, where in the first two cases these equations are real.

4. Numerical results

We will choose the scales $\mathcal{L}$ and $\mathcal{T}$, assuming

$$\mathcal{L} = H, \quad \mathcal{T} = H^2/\nu$$

Then the dimensionless parameters take the form

$$\delta = 1, \quad h = 1, \quad \mathrm{Pr} = \frac{\nu}{\chi}, \quad \mathrm{Ma} = \frac{\alpha_T A H^2}{\rho_0 \chi \nu}, \quad C = \frac{\alpha_0 H}{\rho_0 \nu^2}, \quad \omega = \frac{\tilde{\omega} H^2}{\nu}, \quad Q_0 = \frac{g_0 H^3}{\nu^2}$$

(Q_0 is the Galileo number).

Dispersion equations (3.17)–(3.19) enable us to calculate the relation between the critical values of the parameters, for example, $\mathrm{Ma}(\alpha)$ and $\sigma(\alpha)$, when the remaining parameters are fixed. It is well known that, when investigating parametric interactions in mechanical systems, two cases are of particular interest, namely, high-frequency oscillations

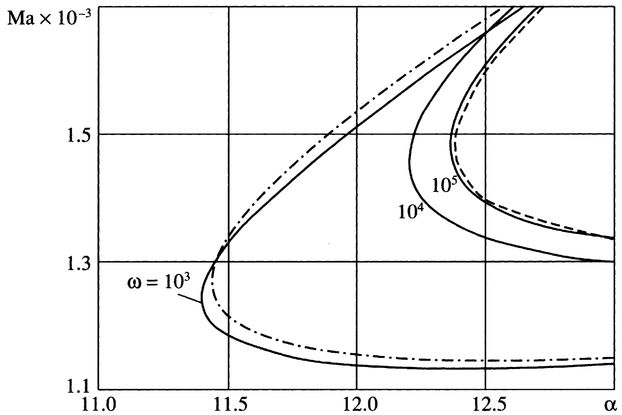

Fig. 1.

and parametric resonances, which will also be considered below. When $\omega \to \infty$ it is assumed that the vibration amplitude is $O(\omega^{-1})$, in which case the amplitude of the vibration rate remains finite, which corresponds to the conditions of the averaging method. The parameter q can be written in the form $q = \sqrt{\mu_s/2}\omega\alpha$, where $\mu_s = \tilde{a}^2h^2/2\nu^2$.[8] The results of calculations were monitored by the arrival at a high-frequency asymptotic form[8] and small wave numbers.[c] During the course of the calculations the number of terms of the continued fraction was determined in such a way that, when a new term was added, the value of the continued fraction changed by no more than 10^{-6}.

4.1. The high-frequency asymptotic form

We have obtained, by calculation, the values of the frequency ω for which the critical parameters Ma(α) and $\sigma(\alpha) = ic(\alpha)$ reach asymptotic values[8] when the remaining parameters are fixed. All the calculations were carried out for a layer with a hard lower wall with Pr = 0.01, $Q_0 = 0$, $C = 10^4$, $\mu_s = 10^4$ and $B = 0$ or $B = \infty$. In the case of monotonic instability ($\sigma = 0$), we obtain that when $\omega = 500$ the values of the Marangoni numbers are identical with the asymptotic values with an accuracy of up to 3%. The results for oscillatory short-wave instability are shown in Fig. 1, where the dash-dot curve corresponds to no vibration, while the dashed curve corresponds to the average problem. It can be seen that vibration can have both a destabilizing effect ($\omega = 10^3$), and a stabilizing effect ($\omega = 10^4$). When $\omega = 10^5$ the graph of Ma(α) differs from the high-frequency asymptotic form by less than 0.5%.

4.2. Resonances

It is well known that when the parameters vary the Floquet exponent σ may be equal to zero (synchronous perturbations), iω/2 (double-period perturbations), or equal to $\pm ic$, $c > 0$ (quasi-periodic perturbations). Since, together with each point σ the points $\sigma + in\omega$, $n \in \mathbb{Z}$ also belong to the Floquet spectrum, all the equalities must be understood as equalities with respect to the modulus $i\omega$. The regions of parametric resonance, corresponding to T-periodic and $2T$-periodic modes ($T = 2\pi/\omega$), were obtained numerically using the transcendental real equations (3.18) and (3.19). The calculations were carried out for $\mu_s = 10^4$, Bi = 0 and $B = 0$. The behaviour of the resonance curves corresponding to $\sigma = i\omega/2$ is shown in Fig. 2 for various values of the frequency ω. The dashed curve corresponds to oscillatory instability, while the continuous branch corresponds to monotonic instability, in the case of the high-frequency asymptotic form. As a result of calculations we obtained that, for the values of the parameters indicated, the resonance neutral curves, which bound the instability regions, are closed and do not disappear as the frequency increases, but are shifted to the right and raised upwards.

The behaviour of the resonance curves Ma(α) and $\sigma = 0$ is interesting (Fig. 3). When $\omega = 1000$ the neutral curve is closed, when $\omega = 1700$ it consists of two closed branches, and when $\omega = 1750$ the neutral curve consists of three closed

[c] Zen'kovskaya S M. Long-wave oscillatory instability of thermocapillary flows in a horizontal layer. Rostov-on-Don, 2005. Deposited at VINITI 9 August 2005, No.1135-V, 2005.

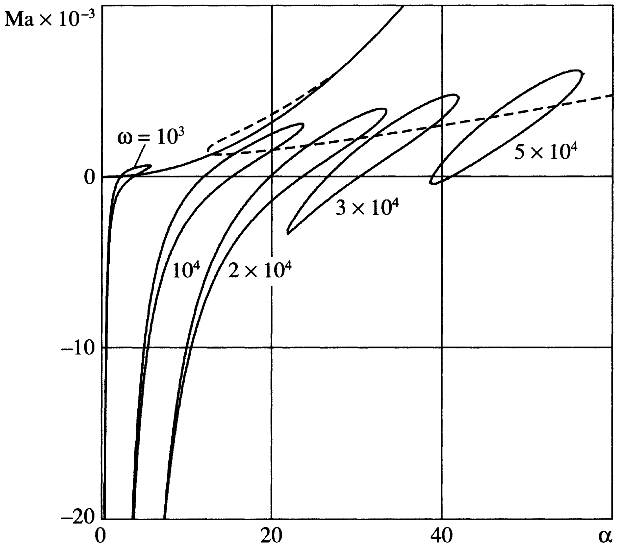

Fig. 2.

branches. A fragment of all the patterns with curves corresponding to $\omega = 1700$, 1750, 2000 and 3000 is shown on an increased scale. As ω increases (for example, when $\omega = 2000$) the branches in the region $\mathrm{Ma} < 0$ disappear, and only the closed curves remain, which, when ω increases, are moved upwards and are shifted to the right. The overall pattern of the behaviour of the resonance curves when $\omega = 10^3$, 5×10^3 and 3×10^4 is shown in Fig. 4. Two types of curves correspond to each value of ω, when $\sigma = i\omega/2$ and $\sigma = 0$. In all cases the instability regions are inside the loop formed by the resonance curves.

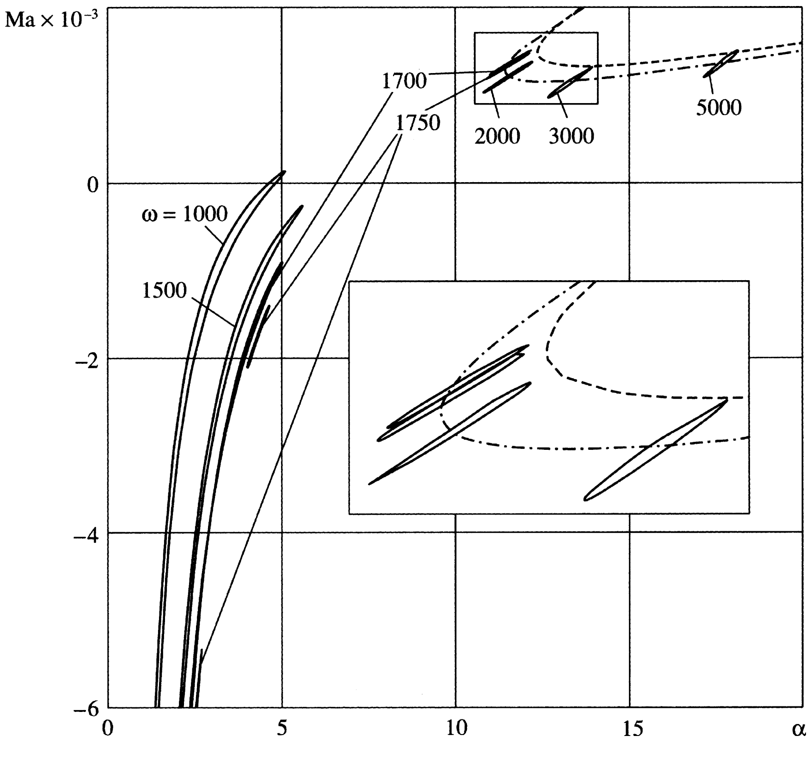

Fig. 3.

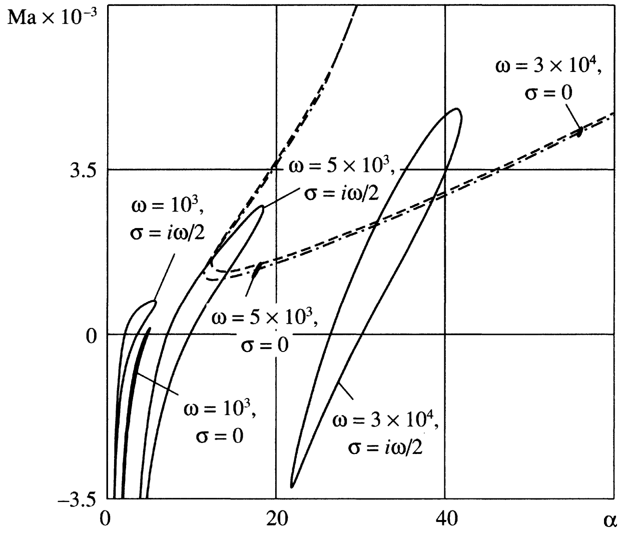

Fig. 4.

5. The long-wave asymptotic form of oscillatory instability

Oscillatory instability in Marangoni convection when there is an isothermal hard wall was first detected numerically,[25] where the critical values of Ma turned out to be negative. Further calculations in Refs. 6–8 and in the present paper showed that, when there are vibrations in the region of small wave numbers α, the critical values of $\mathrm{Ma}(\alpha)$ are also negative, and the principal terms of the asymptotic form are independent of the vibration parameters.

In the preprint listed in the third footnote, in the case of hard and soft walls, for critical values of the parameters $c(\alpha)$ and $\mathrm{Ma}(\alpha)$ as $\alpha \to 0$, the asymptotic was constructed in the form

$$c = c_0 + c_1\alpha^2 + \ldots, \quad \mathrm{Ma} = \mathrm{Ma}_0\alpha^{-2} + \mathrm{Ma}_1 + \ldots \tag{5.1}$$

For c_0 transcendental equations were written, while for the number Ma_0 an explicit formula was given. It was shown that the principal terms of expansions (5.1) depend only on the Prandtl number, the Biot number Bi and B. The case $\mathrm{Bi}=0$ and $B=0$ or $B=\infty$ was investigated in detail. Thus, for example, in the case of an isothermal soft wall ($B=\infty$) and a non-heat conducting hard wall ($B=0$) it was found that the transcendental equation for the unknown $y = \sqrt{2c_0}$ is the same and has the form

$$\mathrm{sh}\sqrt{\mathrm{Pr}}y\sin y - \sin\sqrt{\mathrm{Pr}}y\,\mathrm{sh}\,y = 0 \tag{5.2}$$

Hence the relation $y(\mathrm{Pr}^{-1}) = \sqrt{\mathrm{Pr}}y(\mathrm{Pr})$ follows, from which we obtain that when $\alpha \to 0$ the frequency of neutral oscillations when $\mathrm{Pr} < 1$ is greater than when $\mathrm{Pr} > 1$. Moreover, it was proved that when $\mathrm{Pr} = n^2, n > 1, n \in \mathbb{Z}$, Eq. (5.2) has only real roots $x_k = k\pi, k = \in \mathbb{Z}$. Calculation showed that when $\alpha \to 0$ the values of Ma and c agree well with the asymptotic values.

Acknowledgements

This research was supported financially by the Russian Foundation for Basic Research (05-01-00587) and the Programme for the Support of the Leading Scientific Schools (NSh-5747.2006) and within the framework of European Research Group "Regular and Chaotic Hydrodynamics".

References

1. Zen'kovskaya SM, Simonenko IB. The effect of high-frequency vibration on the onset of convection. *Izv Akad Nauk SSSR MZhG* 1966;(5):51–5.
2. Gershuni GZ, Zhukhovitskii YeM, Nepomnyashchii AA. *The Stability of Convective Flows*. Moscow: Nauka; 1989.

3. Zavarykin MP, Zyuzgin AV, Putin GF. An experimental investigation of parametric thermal convection. *Sbornik Nauch Trudov Perm Univ* 2001;(2):79–96.
4. Birikh RV, Briskman VA, Zuyev AL, Chernatynskii VI, Yakushin VI. The interaction of the thermovibration and the thermocapillary mechanisms of convection. *Izv Ross Akad Nauk MZhG* 1994;(5):107–21.
5. Gershuni GZ, Lyubimov DV, Lyubimova TP, Ru B. Convective flows in a cylindrical liquid zone in a high-frequency vibration field. *Izv Akad Nauk SSSR MZhG* 1994;(5):53–61.
6. Birikh RV, Briskman VA, Bushuyeva SV, Rudakov RN. Thermocapillarity and vibrational instability in a two-layer system with a deformable interface. *Sbornik Nauch Trudov Inst Mekh Splosh Sred UrO Ross Akad Nauk* 2003:21–33.
7. Zenn'kovskaya SM, Shleikel' AL. The effect of high-frequency vibration on the onset of convection in a horizontal liquid layer. *Dokl Ross Akad Nauk* 2002;**382**(5):632–6.
8. Zen'kovskaya SM, Shleikel' AL. The effect of high-frequency vibration on the onset of Marangoni convection in a horizontal liquid layer. *Prikl Mat Mekh* 2002;**66**(4):572–82.
9. Saunders BV, Murray BT, McFadden GB, Coriell SR, Wheeler AA. The effect of gravity modulation on thermosolutal convection in infinite layer of fluids. *Phys of Fluids* 1992;**4**(6):1176–89. Ser. A.
10. Birikh RV, Briskman VA, Velarde MG, Cherepanov AA. The influence of the thermocapillary effect on parametric wave excitation. *Dokl Ross Akad Nauk* 1997;**352**(5):616–9.
11. Gresho PM, Sani RL. The effects of gravity modulation on the stability of a heated fluid layer. *J Fluid Mech* 1970;**40**(Pt 4):783–806.
12. Burde GI. Numerical investigation of convection which occurs in the modulated field of external forces. *Izv Akad Nauk SSSR MZhG* 1970;(2):196–201.
13. Gershuni GZ, Zhukhovitskii YeM, Yurkov YuS. Convective stability in the presence of a periodically varying parameter. *Prik Mat Mekh* 1970;**34**(3):470–80.
14. Gershuni GZ, Keller IO, Smorodin BL. Vibrational-convective instability in zero gravity; finite frequencies. *Dokl Ross Akad Nauk* 1996;**348**(2):194–6.
15. Myznikova BI, Smorodin BL. The convective stability of a horizontal layer of a two-component mixture in the modulated field of external forces. *Izv Ross Akad Nauk MZhG* 2001;(1):3–13.
16. Or AC, Kelly RE. The effects of thermal modulation upon the onset of Marangoni Benard convection. *J Fluid Mech* 2002;**456**:161–82.
17. Markman GS, Yudovich VI. A numerical investigation of the onset of convection in a liquid layer under the action of time-periodic external forces. *Izv Akad Nauk SSSR MZhG* 1972;(3):81–6.
18. Markman GS, Yudovich VI. The occurrence of double-periodic convection modes in the periodic field of external forces. *Zh Prikl Mekh Tekh Fiz* 1972;(6):65–70.
19. Markman GS, Urintsev AL. The parametric excitation of convective motion in a liquid, heated from above. *Izv SKNTs VSh Yestestv Nauki* 1977;**1**:24–7.
20. Yudovich VI. The method of continued fractions in the spectral theory of linear differential operators with periodic coefficients. In: *Jubilee Collection Dedicated to 75th Birthday of V.A. Kakichev*. Novgorod: Novgorod. Gas. Univ.; 2001. p. 2–24.
21. Meshalkin LD, Sinai YaG. Investigation of the stability of the stationary solution of a system of equations of the plane motion of an incompressible viscous fluid. *Prikl Mat Mekh* 1961;**25**(6):1140–3.
22. Yudovich VI. An example of the onset of secondary steady or periodic flow when there is a loss of stability of the laminar flow of a viscous incompressible fluid. *Prikl Mat Mekh* 1965;**29**(3):453–67.
23. Zen'kovskaya SM, Yudovich VI. The method of integro-differential equations and continued fractions in the problem of parametric wave excitation. *Zh Vychisl Mat Mat Fiz* 2004;**44**(4):731–45.
24. Or AC. Onset condition of modulated Rayleigh–Bénard convection at low frequency. *Phys Rev E* 2001;**64**:050201 (R).
25. Takashima M. Surface tension driven instability in a horizontal liquid layer with a deformable free surface. *J Phys Soc Japan* 1981;**50**(8):2745–56.

Translated by R.C.G.